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# Driven Harmonic Oscillator

## APPARATUS

- Computer and interface
- Mechanical vibrator and spring holder
- Stands, etc. to hold vibrator
- Motion sensor
- C-209 spring
- Weight holder and five 100-g mass disks

## INTRODUCTION

This is an experiment in which you will plot the resonance curve of a driven harmonic oscillator. Harmonic oscillation was covered in Physics 6A, so we include a partial review of both the underlying physics and the Pasco Data Studio. We will continue to give fairly detailed instructions for taking data in this first Physics 6B lab. However, this is the last experiment with detailed instructions on setting up Data Studio. Henceforth, it will be assumed that you know how to connect the cables to the computer and to the sensors; how to call up a particular sensor; and how to set up a table, graph, or digit window for data taking.

## THEORY

Hooke's Law for a mass attached to a spring states that  $F = -kx$ , where  $x$  is the displacement of the mass from equilibrium,  $F$  is the restoring force exerted by the spring on the mass, and  $k$  is the (positive) spring constant. If this force causes the mass  $m$  to accelerate, then the equation of motion for the mass is

$$-kx = ma. \tag{1}$$

Substituting for the acceleration  $a = d^2x/dt^2$ , we can rewrite Eq. 1 as

$$-kx = m d^2x/dt^2 \tag{2}$$

or

$$d^2x/dt^2 + \omega_0^2 x = 0, \tag{3}$$

where  $\omega_0 = \sqrt{k/m}$  is called the *resonance angular frequency* of oscillation. Eq. 3 is the differential equation for a *simple harmonic oscillator* with no friction. Its solution includes the sine and cosine functions, since the second derivatives of these functions are proportional to the negatives of the functions. Thus, the solution  $x(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$  satisfies Eq. 3. The parameters  $A$  and  $B$  are two constants which can be determined by the initial conditions of the motion. The natural frequency  $f_0$  of such an oscillator is

$$f_0 = \omega_0/2\pi = (1/2\pi)\sqrt{k/m}. \tag{4}$$

In the simple case described above, the oscillations continue indefinitely. We know, however, that the oscillations of a real mass on a spring eventually decay because of friction. Such behavior is called *damped harmonic motion*. To describe it mathematically, we assume that the frictional force is proportional to the velocity of the mass (which is approximately true with air friction, for example) and add a damping term,  $-b dx/dt$ , to the left side of Eq. 2. Our equation for the damped harmonic oscillator becomes

$$-kx - b dx/dt = m d^2x/dt^2 \quad (5)$$

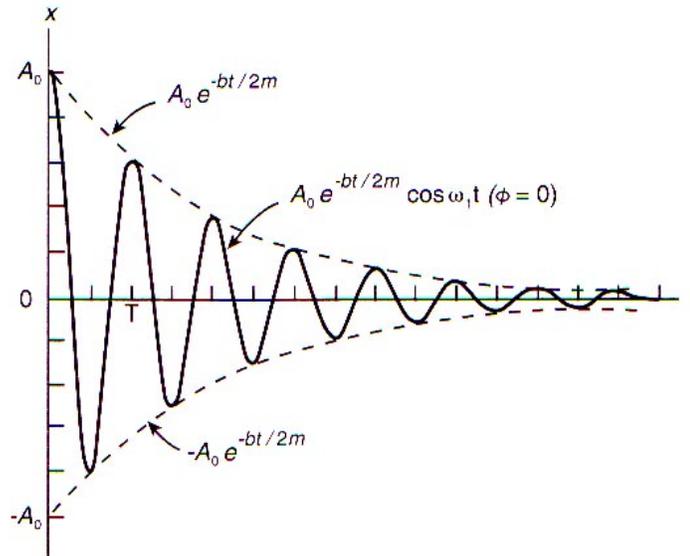
or

$$m d^2x/dt^2 + b dx/dt + kx = 0. \quad (6)$$

Physics texts give us the solution of Eq. 6 (and explain how it is obtained):

$$x(t) = A_0 e^{-bt/2m} \cos(\omega_1 t + \phi). \quad (7)$$

The parameter  $A_0$  is the initial amplitude of the oscillations, and  $\phi$  is the phase angle; these two constants are determined by the initial conditions of the motion. The oscillations decay exponentially in time, as shown in the figure below:



In addition, the angular frequency of oscillation is shifted slightly to

$$\omega_1 = \sqrt{k/m - (b/2m)^2} = \sqrt{\omega_0^2 - (b/2m)^2}. \quad (8)$$

Now imagine that an external force which varies cosinusoidally (or sinusoidally) in time is applied to the mass at an arbitrary angular frequency  $\omega_2$ . The resultant behavior of the mass is known as *driven harmonic motion*. The mass vibrates with a relatively small amplitude, unless the driving angular frequency  $\omega_2$  is near the resonance angular frequency  $\omega_0$ . In this case, the amplitude

becomes very large. If the external force has the form  $F_m \cos(\omega_2 t)$ , then our equation for the driven harmonic oscillator can be written as

$$-kx - b dx/dt + F_m \cos(\omega_2 t) = m d^2x/dt^2 \quad (9)$$

or

$$m d^2x/dt^2 + b dx/dt + kx = F_m \cos(\omega_2 t). \quad (10)$$

The solution of Eq. 10 can also be found in physics texts:

$$x(t) = (F_m/G) \cos(\omega_2 t + \phi), \quad (11)$$

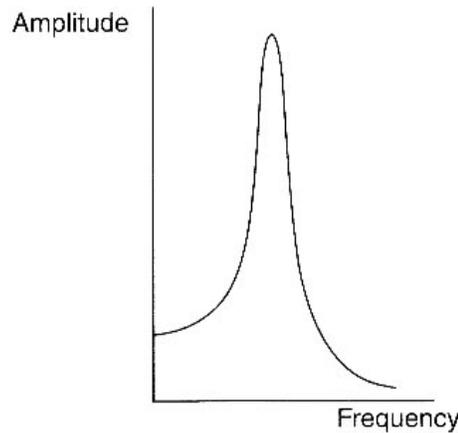
where

$$G = \sqrt{m^2(\omega_2^2 - \omega_0^2)^2 + b^2\omega_2^2} \quad (12)$$

and

$$\phi = \cos^{-1}(b\omega_2/G). \quad (13)$$

The factor  $G$  in the denominator of Eq. 11 determines the shape of the resonance curve, which we wish to measure in this experiment. When the driving angular frequency  $\omega_2$  is close to the resonance angular frequency  $\omega_0$ ,  $G$  is small, and the amplitude of oscillation becomes large. When the driving angular frequency  $\omega_2$  is far from the resonance angular frequency  $\omega_0$ ,  $G$  is large, and the amplitude of oscillation is small.

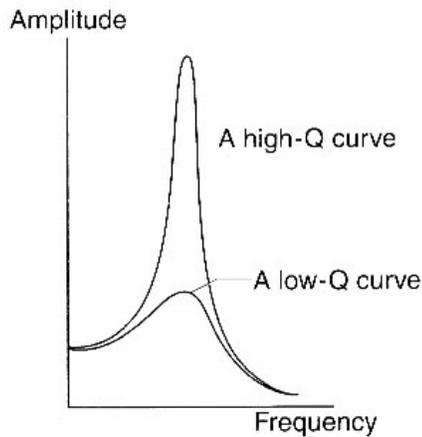
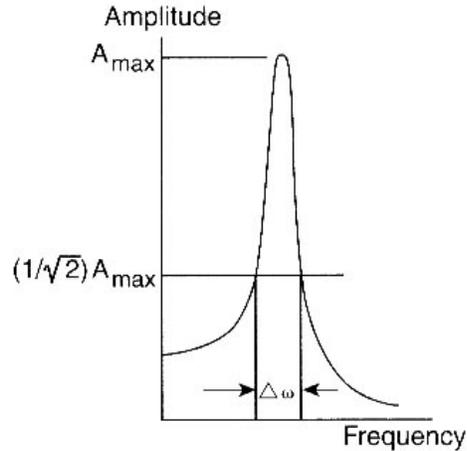


**Figure 2**

This is the curve we wish to measure in the experiment.

## THE QUALITY FACTOR

The sharpness of the resonance curve is determined by the quality factor (or  $Q$  value),  $Q$ . If the frictional force, measured by the parameter  $b$ , is small, then  $Q$  is large, and the resonance curve is sharply peaked. If the frictional force is large, then  $Q$  is small, and the resonance curve is broad (Figure 3).


**Figure 3**

**Figure 4**

The general definition of  $Q$  is

$$Q = 2\pi \times (\text{energy stored}) / (\text{energy dissipated in one cycle}). \quad (14)$$

(If the energy dissipated per cycle is small, then  $Q$  is large, and the resonance curve is sharply peaked.) Physics texts derive the relationship between  $Q$  and the motion parameters,

$$Q = m\omega_1/b, \quad (15)$$

and relate  $Q$  to the sharpness of the resonance peak,

$$Q = \omega_1/\Delta\omega, \quad (16)$$

where  $\Delta\omega = \omega_{\text{high}} - \omega_{\text{low}}$  is the difference in angular frequencies at which the amplitude has dropped to  $1/\sqrt{2}$  of its maximum value (see Figure 4). Note that  $Q$  also controls the exponential damping factor,  $e^{-bt/2m}$ , in Eq. 7 and Figure 1. Using Eq. 15, we can show that

$$e^{-bt/2m} = e^{-\omega_1 t/2Q}. \quad (17)$$

The reciprocal of the factor which multiplies  $t$  in the exponent,  $2m/b = 2Q/\omega_1$ , is the time required for the amplitude of oscillation to decay to  $1/e$  of its initial value (the so-called “e-folding time”).

## EXPERIMENTAL SETUP

This experiment utilizes a signal interface which drives a mechanical vibrator attached to a spring with a mass. We will measure the position of the mass by echo location using the motion sensor (sonic ranger).

1. Plug the yellow-banded cable of the motion sensor into digital channel 1 of the signal interface and the other cable into digital channel 2. From the mechanical vibrator, plug the red and black wires into the output of the signal interface.
2. Turn on the signal interface and the computer.

3. Call up PASCO Capstone. Click on “Hardware Setup” to display the interface. Click on channel 1 of the interface and select “Motion Sensor II”. Click on the yellow circle at the output of the interface. This will add the Output Voltage-Current Sensor.
4. Click on “Signal Generator”. Set Waveform to Sine, set Frequency to 10 Hz, and set Amplitude to 1 V.
5. Click the “On” switch in the signal generator window, and check that the mechanical vibrator stem is shaking. Experiment with the up and down arrows to adjust the frequency and amplitude of the vibrations. You can also click on the number itself and type the desired value. Then click “Off”.

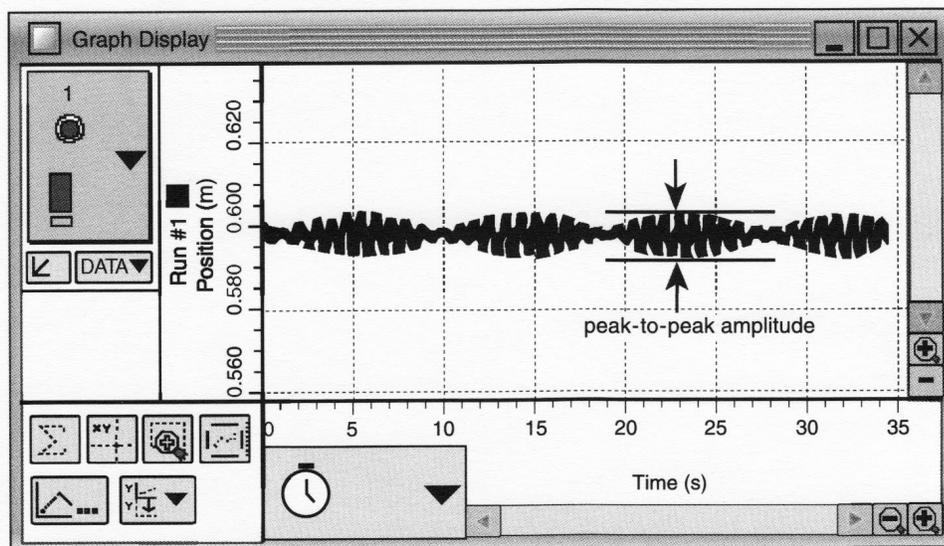
### PROCEDURE PART 1: FINDING THE NATURAL FREQUENCY

1. Attach a C-209 spring with a total mass of 450 g (400-g mass + 50-g mass holder) to the vibrator stem. Place the motion sensor on the floor under the mass-spring system.
2. In the following procedures, be very careful not to drop masses onto the motion sensor. Secure the spring holder firmly to the vibrator stem. If the vibrations become large, they might shake a mass loose. Do not leave masses unattended on the spring; set them aside immediately when you stop taking measurements for a while.
3. Select the “Text & Graph” option on Capstone. Click on the “Select Measurement” button on the  $y$ -axis of the graph. Under Motion Sensor II, click on “position”.
4. Make sure the signal generator switch is “Off” in its window. With your hand, set the mass gently vibrating, click “Record”, then “Stop” after approximately 15 seconds.
5. To zoom in on your data, click on the “Scale axes to show all data” button above the graph (square with a red arrow pointing diagonally).
6. Check that you are obtaining clear oscillations on the graph. If not, adjust the positions of the motion sensor and spring vibrator accordingly. You can delete experimental runs by clicking the drop down arrow next to “Delete Last Run”. Record a clear set of 12 – 15 oscillations.
7. Note a set of 10 clear oscillations on your graph. Record the time at the beginning of the oscillations and the time at the end of 10 complete oscillations. The frequency (number of oscillations per second) is equal to 10 divided by the time required for 10 complete oscillations.
8. Repeat the procedure above three times, and record the average frequency. (This is sometimes called the natural frequency.)

### PROCEDURE PART 2: PLOTTING THE RESONANCE CURVE

In this section, you will verify that resonance occurs when a driving force is applied at the natural frequency of the oscillator.

1. Set the frequency of the signal generator to your measured natural frequency and the amplitude to approximately 1 V. Click the “Auto” box on the signal generator window. This will automatically turn the generator on when the “Record” button is clicked and switch it off when “Stop” is clicked.
2. Set the mass at rest, and click “Start”. Observe the oscillations building up. Make sure they do not get too wild; if so, stop and reduce the amplitude of the signal generator, and start again.
3. Take all your measurements below at the same driving amplitude. First, locate the resonance, set a reasonable driving amplitude, and proceed with measurements on either side of the resonance. (You may find that the oscillations build up and later decay. This is fine. When you are not exactly on resonance, the driven oscillations are “beating” against the natural frequency. At resonance, the amplitude builds up to a large value and remains there. The pattern above is a “transient” which will decay eventually (at least in the ideal case).)



4. Change the frequency by small steps until you locate the resonance exactly, and record the amplitude of oscillation with the frequency in the “Data” section.
5. When taking your measurements, expand the  $y$ -scale by clicking and dragging on the  $y$ -axis, read off the extremes of the oscillations at their maximum amplitude (as in the illustration above), and record the values in the “Data” section. The difference of the two extreme numbers is, of course, the peak-to-peak amplitude of oscillation. Compute this amplitude, and record it in the “Data” section. Be sure your mass is exactly at rest before starting a run.
6. Take measurements at different frequencies until you have a series of measurements (say, 10 or more) which cross the resonance, and continue on either side far enough so the oscillations

are quite small compared to the maximum at resonance. Use small frequency steps with the up and down arrows near the resonance to map it accurately and obtain a nice smooth curve. You can adjust the number of decimal places by clicking on the left and right arrows near the frequency setting.

- When you have a good series of measurements across the resonance, make a careful plot of the resonance curve in Excel. Always title your graph and label the axes. Show your work to the TA. You have completed the required part of the experiment. The next steps result in varying degrees of additional credit.

## DATA

### Procedure Part 1:

- Frequency (Trial 1) = \_\_\_\_\_  
 Frequency (Trial 2) = \_\_\_\_\_  
 Frequency (Trial 3) = \_\_\_\_\_
- Average frequency = \_\_\_\_\_

### Procedure Part 2:

- Amplitude = \_\_\_\_\_  
 Frequency = \_\_\_\_\_
- Maximum position = \_\_\_\_\_  
 Minimum position = \_\_\_\_\_  
 Peak-to-peak amplitude = \_\_\_\_\_
- Maximum positions = \_\_\_\_\_  
 \_\_\_\_\_  
 \_\_\_\_\_  
 \_\_\_\_\_  
 \_\_\_\_\_  
 \_\_\_\_\_

Minimum positions = \_\_\_\_\_

Peak-to-peak amplitudes = \_\_\_\_\_

8. Plot the resonance curve using Excel. Remember to label the axes and title the graph.

## ADDITIONAL CREDIT PART 1: FINDING THE SPRING CONSTANT (2 mills)

We determined the spring constant  $k$  for springs several times in the Physics 6A lab by using a meter stick and a set of masses, entering the measurements into Excel, and finding the slope of the line.

Here is an alternate method. Use the motion sensor to take the position measurements. Get rid of any previous data runs by clicking the drop down arrow under “Delete Last Run” and clicking “Delete All Runs”. Drag a table symbol over to your work space. Click the “Select Measurement” button and choose “Position”. Make sure that the Signal Generator is in the “Off” setting and not “Auto”.

Change the recording mode from “Continuous Mode” to “Keep Mode” at the bottom of the screen. Start with only one of the five masses on the hanger. With the spring system still directly above the motion sensor, click “Preview”. With the mass at rest, click the “Keep Sample” button. Add another mass and click the “Keep Sample” button again. Do this until you have values recorded for 5 different masses. Click the “Stop” button.

Use these 5 values of position and the five masses you used to make a plot of force vs displacement. Use Excel to do a best-fit line to your data and determine the slope. How is the slope of your graph related to the spring constant? You will need to use the correct units of force (Newtons) to get  $k$  in the proper units. You could either convert each mass entry into units of force and make your keyboard entries in Newtons, or just proceed with mass in units of kilograms and make the conversion of units at the end when you read the slope from the curve-fitting routine. In your work, make clear which method you are using.

Slope of line = \_\_\_\_\_

Spring constant  $k$  = \_\_\_\_\_

## ADDITIONAL CREDIT PART 2: PREDICTING THE RESONANCE FREQUENCY (1 mill)

If there were no friction, the resonance frequency would be

$$f_0 = \omega_0/2\pi = (1/2\pi)\sqrt{k/m}. \quad (4)$$

Compute this value of  $f_0$ , and compare it with your measured value of the resonance frequency. Record your results clearly.

$f_0$  (computed from Eq. 4) = \_\_\_\_\_

$f_0$  (measured) = \_\_\_\_\_

Percentage difference in  $f_0$  = \_\_\_\_\_

**ADDITIONAL CREDIT PART 3: THE QUALITY FACTOR AND FRICTION  
(2 mills)**

We saw that

$$Q = \omega_1 / \Delta\omega, \tag{16}$$

where  $\Delta\omega = \omega_{\text{high}} - \omega_{\text{low}}$ . Compute the value of  $Q$  for your mass on a spring by finding the angular frequencies on your resonance curve where the amplitude has dropped to  $1/\sqrt{2}$  of its maximum value.

$Q$  is also equal to  $m\omega_1/b$ , by Eq. 15. By approximating  $\omega_1$  as  $\omega_0$ , estimate the value of  $b$ . Then compute the true resonance angular frequency, including friction, from

$$\omega_1 = \sqrt{k/m - (b/2m)^2}. \tag{8}$$

What is the percentage difference between  $\omega_1$  and  $\omega_0$ ? Could your measurements of the resonance curve have distinguished between  $\omega_1$  and  $\omega_0$ ?

$\omega_{\text{high}} =$  \_\_\_\_\_

$\omega_{\text{low}} =$  \_\_\_\_\_

$Q =$  \_\_\_\_\_

$b =$  \_\_\_\_\_

$\omega_1 =$  \_\_\_\_\_

Percentage difference between  $\omega_1$  and  $\omega_0 =$  \_\_\_\_\_